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ALGORITHMS FOR TRACKING A SOUND SOURCE

USING FREQUENCY AND BEARINGS

C. Kent

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USING FREQUENCY AND BEARINGS

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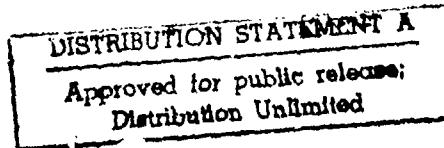
RESUME

Dans ce rapport, on développe certains fondements théoriques d'algorithmes conçus pour permettre à une ou plusieurs unités mobiles de recherche de poursuivre une source de bruit qui se déplace en ligne droite. Les unités de recherche mesurent le gisement et la fréquence apparente afin d'estimer la distance, la fréquence réelle et les composantes de la vitesse de la source. On présume que les mesures de gisement et de fréquence sont sujettes à des erreurs aléatoires. (NC)

ABSTRACT

This report develops some of the analytical support for algorithms designed to permit one or more mobile searchers to track a sound source moving in a straight line. The searchers measure bearing and received frequency to infer range, the true frequency and velocity components. The measurements of bearing and frequency are subject to random errors. (U)

TABLE OF CONTENTS

1.0 INTRODUCTION

This report presents a basic numerical treatment of the two-dimensional problem posed by the passive tracking of a sound source by one or more searchers that measure the bearing and frequency of the source. The question is: can the position, the course, the speed, and the frequency of the source be determined with usable accuracy when the observations about its bearings and its frequency are made over a period of time and in the presence of error noise?

Constant source velocity, i.e. constant speed and direction, is the only case considered in this report. Further, the measurement errors of both bearing and frequency are assumed to be normally distributed, with specified means and standard deviations and to be independent from measurement to measurement.

Clearly, complete tracking cannot always be accomplished on the basis of frequency observations alone, because a source moving along a tangent to a circle around a single searcher will always exhibit the same Doppler effect, no matter which tangent this source follows. Of course, the use of multiple searchers or changes in searcher velocity during measurement may yield additional information, but the algorithms discussed in this report are based on a single fundamental frequency relation, which uses both bearing and frequency information.

There seems to be some confusion about the exact form of the Doppler equation, so I first related the usual approximation for Doppler frequency to a correct relation for observed frequency when both the source and the searcher are moving relative to the propagation medium; I

then numerically explored the accuracy of the approximation. This is reported in Section 2.0.

In Section 3.0, the acceptable approximation relation found in Section 2.0 is used to develop a fundamental system of linear equations between the source parameters and the measured data. Each measurement provides one equation, and it is assumed that the total number of measurements, N, exceeds three, which is the number of parameters to be determined. The solution of the resulting system of equations, in least squares sense, for f, $\dot{X}_T$, $\dot{Y}_T$, is discussed and the effects of constraints on the region of acceptable solutions are considered in Section 3.0. Since the linear equations of Section 3.0 do not explicitly contain range, their solution produces no information about this parameter.

Several ways to introduce range as a parameter in the linear system are discussed in Sec. 4.0. The most informative one seems to be the straightforward grid search for a least squares minimum deviation between predicted and observed frequency-only data. However, the grid search, which uses an internal iteration, is very time consuming, and can be taken only as indicative of possible information in the data. This information might then be better extracted by some other procedure. In Section 5.0 comments are offered on sample calculations made with the algorithms developed earlier. The actual results of those calculations are not given. It is worth commenting that the problem has a unique solution even on the basis of only three measurements, provided these measurements are free of errors. Of course, there are time-honored methods for passive tracking using bearings only, but they require more observations, multiple observers, or maneuvers by the observers. The

proof that the joint bearing-frequency problem has a unique solution for three observations is given in Appendix B.

In Appendix A the APL implementations of the algorithms outlined in the text are detailed. Much improved implementations could be produced, but those given are adequate to test the algorithms. What is really needed is a good algorithm for determining range when this parameter is not available as a primary variable.

The text also includes some comments on the effect of providing estimates of one or more parameter values to decrease the breadth of the search.

This work was performed at DREV in the first half of 1980, under PCN 32D37, Tactical Towed Array Study.

2.0 THE FREQUENCY SHIFT RELATION AND ITS APPROXIMATION

When both the source of a sound and the receiver are in motion with respect to the propagation medium, a component of the resultant frequency shift arises from each of the two motions. In the following analysis, the absolute frequency of the source is represented by the accompanying absolute time between wave crests, or pulses, denoted by Δ. Frequency is the reciprocal of Δ. Thus, absolute percentage errors in Δ and in frequency are the same, since

$$f = 1/\Delta$$

$$Df = -D\Delta/\Delta^2$$

$$Df/f = -D\Delta/\Delta$$

In Fig. 1, T and T' are the positions of the source, a time Δ apart, and S and S' those of the receiver when the pulses emitted at T and T' are received. The time interval between the reception of the two pulses is denoted Δ_a. The rest of the notation used in the diagram is defined as follows:

$\vec{V}_T$ = the vector velocity of the source with respect to the medium

V_T = the speed of the source, or magnitude of $\vec{V}_T$

C = the speed of sound in the medium

$\vec{V}_S$ = the vector velocity of the receiver with respect to the medium

V_S = the speed of the receiver, or magnitude of $\vec{V}_S$

$\vec{R}_1$ = the vector from S to T, i.e. the reverse of the propagation path for the first pulse

R_1 = the distance from S to T
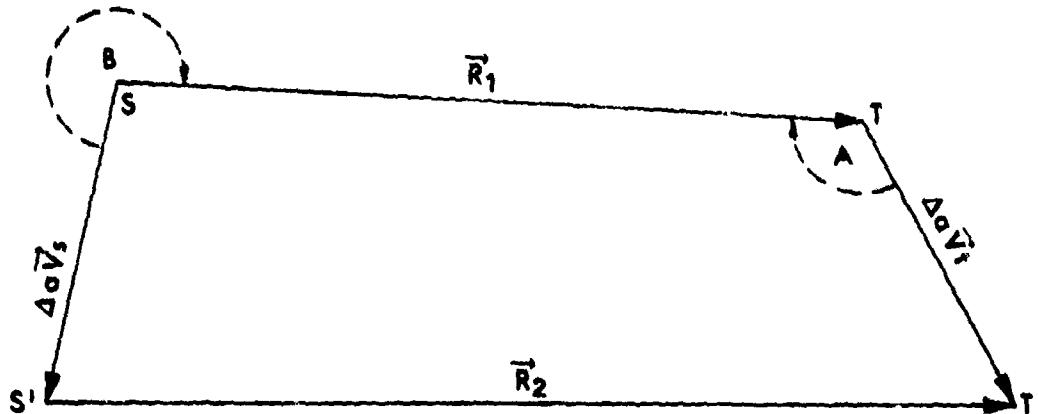
Figure 1 – Geometry of bearing and frequency measurements

$\vec{R}_2$ = the vector from S' to T', i.e. the reverse propagation path
for the second pulse

R_2 = the distance from S' to T'

A and B are angles measured clockwise from the heading vectors to the propagation paths, as illustrated.

To compare the time interval between the reception of the pulses, Δ_a, with that between their emission, Δ, we write an equation stating that time along path T + T' + S' must equal time along path T + S + S'.

$$\Delta + R_2/C = \Delta_a + R_1/C \qquad [1]$$

If we introduce coordinates $\langle X_T, Y_T \rangle$ and $\langle X_S, Y_S \rangle$ to describe positions of the points T and S, and let

$$X = X_T - X_S \qquad Y = Y_T - Y_S \qquad [2]$$

then, of course,

$$R_1 = (X^2 + Y^2)^{\frac{1}{2}} \tag{3}$$

and

$$R_2 = ((X + \Delta\dot{X}_T - \Delta_a\dot{X}_S)^2 + (Y + \Delta\dot{Y}_T - \Delta_a\dot{Y}_S)^2)^{\frac{1}{2}} \tag{4}$$

where the dots indicate time derivatives, or velocity components. A direct substitution of [3] and [4] into [1] yields:

$$\Delta_a = \Delta + \frac{1}{C}(R_2 - R_1)$$

$$= \Delta + \frac{1}{C}(((X + \Delta\dot{X}_T - \Delta_a\dot{X}_S)^2 + (Y + \Delta\dot{Y}_T - \Delta_a\dot{Y}_S)^2)^{\frac{1}{2}} - (X^2 + Y^2)^{\frac{1}{2}}) \tag{5}$$

Equation [5] is an implicit relationship for the unknown, Δ_a, in terms of the other quantities, all of which are known. This equation cannot be easily solved directly for Δ_a, but that step is unnecessary since [5] can be used as an iteration equation which converges very rapidly to the value of Δ_a, starting from $\Delta_a = \Delta$.

The APL program DEXACT (Appendix A) is an iteration which serves to calculate Δ_a to accuracy 10^{-10}.

The iteration based upon [5] converges to the true value of Δ_a, but it is not in the usual form of the Doppler equation, and a closed-form approximation which provides sufficient accuracy would be more useful. To derive the best linear approximation to [5], one regards the quantity in brackets, $R_2 - R_1$, for fixed Δ and Δ_a, as a function of the six variables $X, Y, \dot{X}_T, \dot{Y}_T, \dot{X}_S, \dot{Y}_S$. As such, it can be

expanded in a six-dimensional Taylor expansion about point X,Y,0,0,0,0. If this is done, and if only the first-order (linear) terms are retained, the approximated equality [6] results:

$$\Delta_a = \Delta + \frac{1}{C} \left(\frac{X}{R} \Delta \dot{X}_T + \frac{Y}{R} \Delta \dot{Y}_T - \frac{X}{R} \Delta_a \dot{X}_S - \frac{Y}{R} \Delta_a \dot{Y}_S \right) \qquad [6]$$

Noting that X/R and Y/R are the components of the unit vector $\vec{r}_1 = \vec{R}_1/R_1$, the dot or scaler product of two vectors can be used to give [6] a more symmetric form.

$$\frac{\Delta_a}{\Delta} = \frac{1 + \frac{1}{C} \vec{r}_1 \cdot \vec{V}_T}{1 + \frac{1}{C} \vec{r}_1 \cdot \vec{V}_S} \qquad [7]$$

The Doppler shift is usually represented by [7].

The dot products in [7] can be written in terms of angles A and B:

$$\vec{r}_1 \cdot \vec{V}_T = V_T \cos (\pi - A) = -V_T \cos A$$

$$\vec{r}_1 \cdot \vec{V}_S = V_S \cos (2\pi - B) = V_S \cos B$$

and [7] then becomes

$$\frac{\Delta_a}{\Delta} = \frac{1 - \frac{1}{C} V_T \cos A}{1 + \frac{1}{C} V_S \cos B} \qquad [8]$$

A better approximation to equation [5] can be obtained with the expression generally used for Doppler shift, particularly in the case of one-dimensional motion:

$$\frac{\Delta_a}{\Delta} = 1 + \frac{\dot{R}}{C} \qquad [9]$$

where $\dot{R}$ is the time derivative of the distance between the receiver and the source. From Fig. 1, it can be seen that

$$\dot{R} = V_T \cos (\pi - A) - V_S \cos (2\pi - B)$$
$$= -V_T \cos A - V_S \cos B$$

and hence, [9] can be written

$$\frac{\Delta_a}{\Delta} = 1 - \frac{1}{C} (V_T \cos A + V_S \cos B) \qquad [9']$$

But [9'] also results from [8] because $(V_S/C)\cos B$ is small with respect to 1; hence,

$$(1 + \frac{1}{C} V_S \cos B)^{-1} = 1 - \frac{1}{C} V_S \cos B$$

approximately. If the term involving $1/C^2$ is discarded in

$$\frac{\Delta_a}{\Delta} = (1 - \frac{1}{C} V_T \cos A) (1 - \frac{1}{C} V_S \cos B) \qquad [10]$$

when the right side is expanded, then [9] can be seen as an approximation to [8].

The errors inherent in these approximations were sampled by the APL function DAPPROX (see Appendix A) for two signal frequencies nearly two octaves apart. The results were not much different for the two frequencies. The errors of approximation of [5] by [8] and [9] were of the order of 10^{-5} percent and 10^{-3} percent, respectively.

It should be stressed that these errors arise from an approximation formula and are therefore not only small, but also systematic. They are not random errors. For frequency tracking, where the true frequency is unknown, systematic errors do not influence the accuracy so long as they do not change significantly while tracking data are gathered. Further numerical investigation of the errors involved in using [7] to compute the Doppler shift reveals that they are systematic and slowly varying. Thus, it is concluded that equation [7]:

$$\frac{\Delta_a}{\Delta} = \frac{1 + \frac{1}{C}\vec{r}_1 \cdot \vec{V}_T}{1 + \frac{1}{C}\vec{r}_1 \cdot \vec{V}_S}$$

is an acceptable approximation for use in the frequency tracking problem.

3.0 THE FUNDAMENTAL LINEAR SYSTEM

The basic equation for tracking a sound-emitting source by measuring its bearing and frequency continuously, or at several discrete points in time is:

$$\frac{f_a}{f} = \frac{C + \vec{r} \cdot \vec{V}_S}{C + \vec{r} \cdot \vec{V}_T} \qquad [11]$$

as derived in the preceding section, where:

f_a = apparent frequency to the listener

f = true frequency

C = speed of sound in the medium

$\vec{r}$ = a unit vector pointing along the reverse of the pulse propagation path, from searcher to source

$\vec{V}_S$ = velocity vector of the searcher, relative to the medium

$\vec{V}_T$ = velocity vector of the source, relative to the medium.

It should be noted that, because $\vec{r}$ is a unit vector, the range from searcher to source does not explicitly appear in [11]. Because of this, range is a secondary parameter in the search problem, and is difficult to determine since the measured quantities of frequency and bearing are not sensitive to range variation. In the search algorithms proposed here, range is not initially determined, but it is left to be determined by a follow-up algorithm.

If B is the value of the bearing angle, measured clockwise from
the north, then the unit vector r has components

$$< \cos B, \sin B >$$ [12]

In this treatment the positive X-axis is oriented toward the
north and the positive Y-axis toward the east. Using the angular
representation [12] for r, [11] may easily be written as:

$$\dot{X}_T \cos B + \dot{Y}_T \sin B - f \left(\frac{C + \vec{r}.\vec{V}_S}{f_a} \right) = -C$$ [13]

By so writing [11], we see that each measurement of B and f_a, together
with the known values of C and V_S, determines a set of coefficients

$$\cos B, \sin B, - \frac{C + \vec{r} . \vec{V}_S}{f_a}, C$$

for a linear equation in the three unknowns $\dot{X}_T$, $\dot{Y}_T$, f.

Thus, taking bearing and frequency measurements simultaneously at
N different moments results in a system of N simultaneous linear
equations in the three unknowns.

Three such equations would generally suffice to determine values
for $\dot{X}_T$, $\dot{Y}_T$, f, and the remaining N-3 equations would either complement
the solution with redundant information, or else be inconsistent.   Note
that in seeking fixed solutions $\dot{X}_T$, $\dot{Y}_T$ and f, we implicitly assume that

during the measurement period, the source changes neither velocity nor frequency.

The usual situation, measurements with error components (noise), will normally give rise to an overdetermined or inconsistent system of equations of the form [13]. It is possible, in this case to use the generalized inverse of the matrix M, below, to obtain the least squares solution to the system:

$$\dot{X}_T \cos B_1 + \dot{Y}_T \sin B_1 - f\left(\frac{C + \vec{r} \cdot \vec{V}_{S_1}}{f_{a_1}}\right) = -C$$

$$\dot{X}_T \cos B_2 + \dot{Y}_T \sin B_2 - f\left(\frac{C + \vec{r} \cdot \vec{V}_{S_2}}{f_{a_2}}\right) = -C \qquad [14]$$

$$\dot{X}_T \cos B_N + \dot{Y}_T \sin B_N - f\left(\frac{C + \vec{r} \cdot \vec{V}_{S_N}}{f_{a_N}}\right) = -C$$

or

$$M \times \begin{pmatrix} \dot{X}_T \\ \dot{Y}_T \\ f \end{pmatrix} = \begin{pmatrix} -C \\ -C \\ \cdot \\ \cdot \\ -C \end{pmatrix} \qquad [14']$$

where [14'] is the matrix equation counterpart of the system [14].

The theory of the generalized inverse can be found in Ref. 1. For example, in the APL language used for the programs listed in this report, the operation ⊞M always yields the generalized inverse of M, unless M has fewer than 3 (in this case) independent columns. Thus, where ⊞ is used, the generalized inverse always results; it is identical with the true inverse where one exists.

Solving the system [14] through the generalized inverse as in the first part of the APL function DOPPVEC2 (Appendix A) yields an unconstrained least squares solution for source velocity and frequency. However, the search problem is seldom unconstrained. Several types of constrained problems and their solutions are discussed in the next section.

3.1 Constraints

The first type of constraint provided by additional (outside) information is easily disposed of. If frequency is known, velocity components are found by solving system [14], as we discussed above, and this operation is simplified because there are only two unknowns left, instead of three. The known frequency constraint is not described further.

The second type of constraint, that of known direction of source movement, without considering speed, is also easy to handle. When the direction is known, $\dot{Y}_T$ becomes a constant multiple of $\dot{X}_T$, and can be replaced in system [14], leading again to a system to be solved for two unknowns. If direction is known to lie within a certain angular sector, the constrained solution is easily obtained from the unconstrained

solution as discussed under the next case.

The third type of constraint, that of known or bounded source speed, will be carefully treated for its solution contains the methods for handling the other constraints mentioned above.

3.2 Constrained Speed

The fundamental linear system [14], which can usually be satisfied only approximately in least squares sense, can be thought of in the following way. Consider the left side of each equation as defining a coordinate function, Z_i, of $\dot{X}_T$, $\dot{Y}_T$, f, as follows:

$$D_i \, (\dot{X}_T, \dot{Y}_T, f) = \dot{X}_T \cos B_i + \dot{Y}_T \sin B_i - f \left(\frac{C + \vec{r} \cdot \vec{V}_{S_i}}{f_{a_i}} \right) = Z_i \qquad [15]$$

If there are N measurements, i varies from 1 to N, and [15] yields N coordinate functions, which define a mapping of the three-dimensional parameter space $\dot{X}_T$, $\dot{Y}_T$, f, into an N-dimensional space. The least squares solution of system [14] can be considered from a geometrical point of view.

The least squares solution vector, $\langle \dot{X}_T^0, \dot{Y}_T^0, f^0 \rangle$ (or point in the three-dimensional space), determines a vector $\langle Z_1^0, Z_2^0, \ldots, Z_N^0 \rangle$ which is as close as possible to the constant vector $\langle -C, -C, \ldots, -C \rangle$, of N coordinates. If the minimum achievable value of distance between $\langle Z_1^0, \ldots, Z_N^0 \rangle$ and $\langle -C, \ldots, -C \rangle$ is denoted S_0, i.e. the least squares

difference is S_0, then the point $\langle Z_1^0, \ldots, Z_N^0 \rangle$ lies on the surface of an N-dimensional sphere with its center at $\langle -C, \ldots, -C \rangle$.

Point $\langle Z_1^0, \ldots, Z_N^0 \rangle$, at distance S_0 from $\langle -C, \ldots, -C \rangle$ might not be attained because of constraints on one or more of $\dot{X}_T$, $\dot{Y}_T$, f. The point at distance S_0 is the unconstrained minimum; if it is unattainable, the algorithm should seek a solution in the constraint region of the three-dimensional space of the parameters $\dot{X}_T$, $\dot{Y}_T$, f which yields a value $\langle Z_1, \ldots, Z_N \rangle$ as close as possible to $\langle -C, \ldots, -C \rangle$, i.e. a constrained minimum. Thus, the algorithm must seek a parameter point $\langle \dot{X}_T^1, \dot{Y}_T^1, f^1 \rangle$ for which the associated $\langle Z_1^1, Z_2^1, \ldots, Z_N^1 \rangle$ lies on the surface of some other sphere about $\langle -C, \ldots, -C \rangle$, of radius $S_1 > S_0$. The equation of the sphere of radius S_1 is:

$$\sum_{i=1}^{N} (Z_i - (-C))^2 = S_1^2$$

or

$$\sum_{i=1}^{N} (\dot{X}_T \cos B_i + \dot{Y}_T \sin B_i - P_i f + C)^2 = S_1^2 \qquad [16]$$

where

$$P_i = \frac{C + \vec{r} \cdot \vec{V}_{S_i}}{f_{a_i}}$$

Squaring the left side of [16] yields

$$A_{11}\dot{X}_T^2 + A_{22}\dot{Y}_T^2 + A_{33}f^2 + 2A_{12}\dot{X}_T\dot{Y}_T - 2A_{13}\dot{X}_Tf$$
$$+ 2CA_1\dot{X}_T - 2A_{23}\dot{Y}_Tf + 2CA_2\dot{Y}_T - 2CA_3f + E = 0$$

[17]

where

$$A_{11} = \sum_{i=1}^{N} \cos^2 B_i, \quad A_{22} = \sum_{i=1}^{N} \sin^2 B_i, \quad A_{33} = \sum_{i=1}^{N} P_i^2$$

$$A_{12} = \Sigma \cos B_i \sin B_i, \quad A_{13} = \Sigma P_i \cos B_i, \quad A_{23} = \Sigma P_i \sin B_i$$

$$A_1 = \Sigma \cos B_i, \quad A_2 = \Sigma \sin B_i, \quad A_3 = \Sigma P_i$$

$$E = NC^2 - S_1^2$$

Equation [17] represents a family of ellipsoids centered on the point $\langle \dot{X}_T^0, \dot{Y}_T^0, f^0 \rangle$. The family parameter S_1 determines the size of each ellipsoid. For $S_1 = S_0$ there is the point ellipsoid at $\langle \dot{X}_T^0, \dot{Y}_T^0, f^0 \rangle$. Figure 2 shows the geometry described above.

In the three-dimensional space of $\dot{X}_T$, $\dot{Y}_T$, f, a constraint of constant speed, V, is described by the surface of a cylinder

$$\dot{X}_T^2 + \dot{Y}_T^2 = V^2$$

and a constraint of bounded speed by the surface and interior of this cylinder,

$$\dot{X}_T^2 + \dot{Y}_T^2 \leq V^2.$$
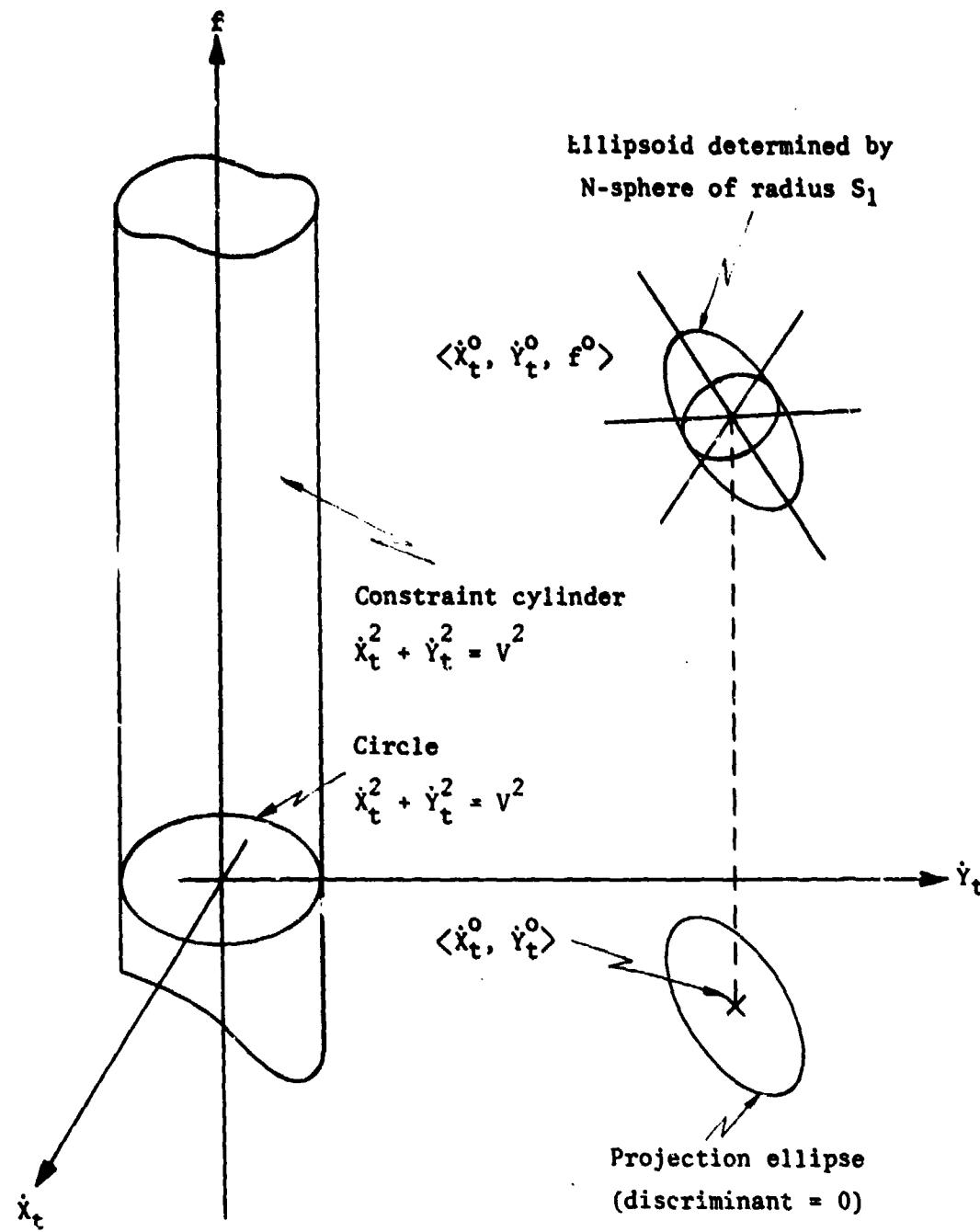
Figure 2 – Geometry of speed constraint

For those constraints, the constrained minimum is obtained when the solution ellipsoids are expanded, by increasing S_1, until tangency is achieved with the constraint cylinder. Of course, if the constraint region consists of the cylinder and its interior, and the unconstrained minimum $\dot{X}_T^0$, $\dot{Y}_T^0$, f^0 lies in the cylinder, then $S_1 = S_0$.

The mathematical problem of finding the point of tangency between ellipsoid and cylinder can be reduced to two dimensions, for this tangency occurs exactly when tangency occurs in the $\dot{X}_T$-$\dot{Y}_T$ plane between the circle $\dot{X}_T^2 + \dot{Y}_T^2 = V^2$ and the ellipse which is the ellipsoid's projection (shadow) in that plane.

The projection of the ellipsoid is describable as the set of points $\dot{X}_T$, $\dot{Y}_T$ above which exactly one point on the ellipsoid can be found. Equivalently, if [17] is regarded as ar equation to be solved for f, given $\dot{X}_T$, $\dot{Y}_T$, the projection points are those for which the two quadratic solutions for f coincide. In other words, they are just the points $\dot{X}_T$, $\dot{Y}_T$ for which tne discriminant of [17], regarded as a quadratic in f, vanishes. Setting that discriminant equal to zero gives:

$$(A_{13}\dot{X}_T + A_{23}\dot{Y}_T + CA_3)^2 -$$
$$A_{33}(A_{11}\dot{X}_T^2 + A_{22}\dot{Y}_T^2 + 2A_{12}\dot{X}_T\dot{Y}_T + 2CA_1\dot{X}_T + 2CA_2\dot{Y}_T + E) = 0$$

or

$$B_{11}\dot{X}_T^2 + B_{22}\dot{Y}_T^2 + 2B_{12}\dot{X}_T\dot{Y}_T + 2B_1\dot{X}_T + 2B_2\dot{Y}_T + F = 0 \qquad [18]$$

where

$$B_{11} = A_{13}^2 - A_{11} A_{33}, \qquad B_{22} = A_{23}^2 - A_{22} A_{33}$$

$$B_1 = C (A_{13}A_3 - A_{33}A_1), \quad B_2 = C (A_{23}A_3 - A_{33}A_2)$$

$$B_{12} = (A_{13}A_{23} - A_{33}A_{12}), \quad F = C^2 A_3^2 - A_{33}E$$

Equation [18] is the ellipse equation to be solved for tangency with the circle equation $\dot{X}_T^2 + \dot{Y}_T^2 = V^2$. Finally, to ensure tangency, another equation can be added by differentiating the circle equation and [18], and then equating slopes at the solution point. If this is done, the following system of equations in three unknowns must be solved. Recall that F is a variable since it contains the radius S_1 which must be increased to produce tangency. S_1 appears nowhere else.

$$B_{11}\dot{X}_T^2 + B_{22}\dot{Y}_T^2 + 2B_{12}\dot{X}_T\dot{Y}_T + 2B_1\dot{X}_T + 2B_2\dot{Y}_T + F = 0$$
$$\dot{X}_T^2 + \dot{Y}_T^2 \qquad\qquad\qquad - V^2 = 0 \qquad\qquad [19]$$
$$B_{12}\dot{X}_T^2 - B_{12}\dot{Y}_T^2 + (B_{22} - B_{11}) \dot{X}_T\dot{Y}_T + B_2\dot{X}_T - B_1\dot{Y}_T = 0$$

The system [19] is solved by a Newton-Raphson iteration in the APL function DOPPVEC2. When the $\dot{X}_T^1$, $\dot{Y}_T^1$, S_1 coordinates of the point of tangency are found, the value of f is recalculated using the ellipsoid [17]. The result of the calculation is stored in an APL vector called STV, which consists of f_T^1, $\dot{X}_T^1$, $\dot{Y}_T^1$.

Figure 3 – Geometry of direction constraint
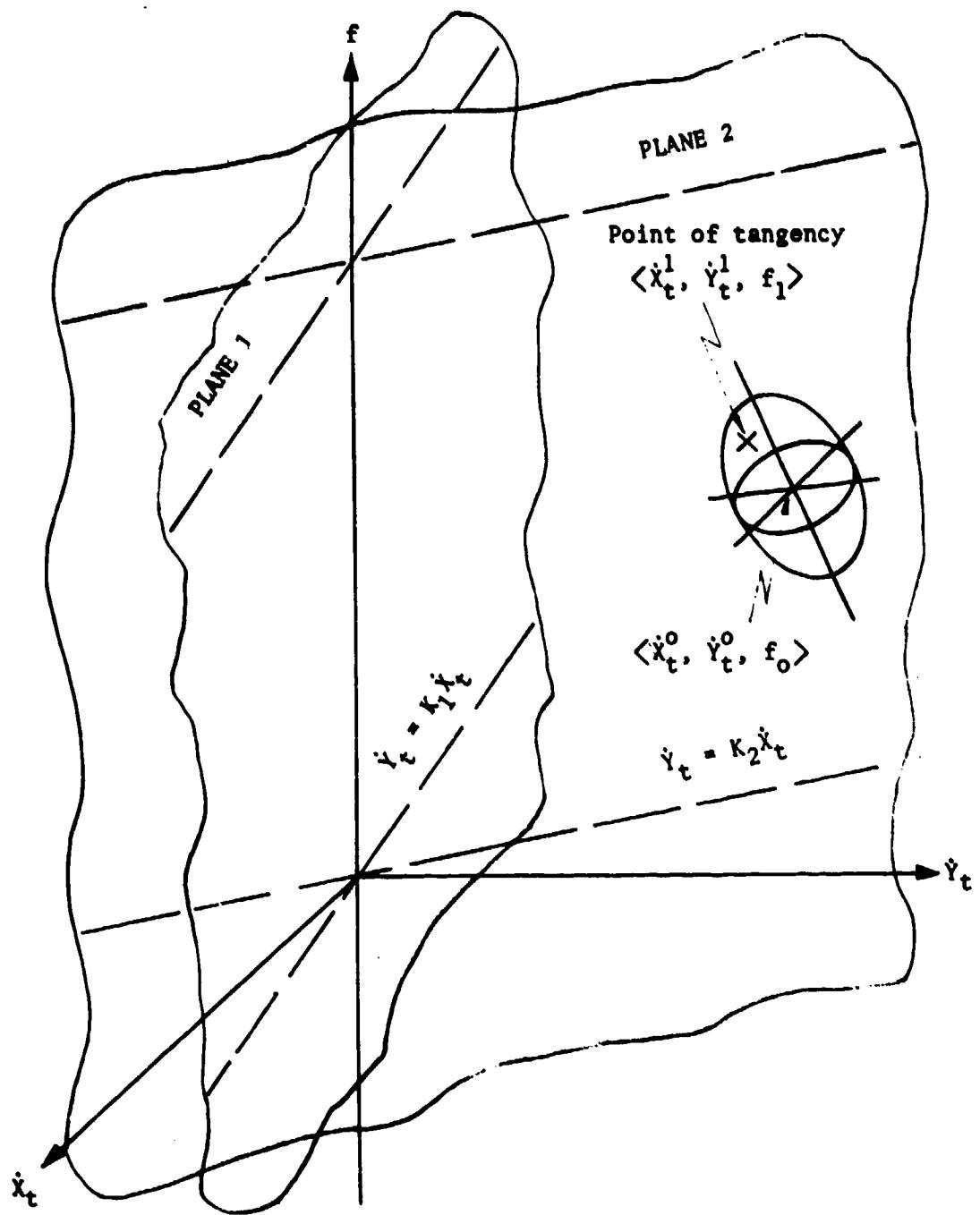
This completes the description for a cylindrical speed constraint.

3.3 Constrained Direction

To solve the constrained direction problem discussed earlier, we must only reconsider the geometry of the constrained minimum problem, as in Fig. 3. In those terms, a direction constraint region consists of a (double) wedge with interior, parallel to the f-axis. The boundary planes of the constraint region are defined by limiting direction lines.

Again, if the unconstrained minimum solution $\langle \dot{X}_T^0, \dot{Y}_T^0, f^0 \rangle$ lies within the constraint wedge, the constrained problem is solved. If not, the situation is identical to the one discussed above, and S_1 must be increased until an ellipsoid becomes tangent to the constraint wedge. However, it is not necessary to repeat the process used for the cylinder constraint region. It will be noted that the desired point of tangency is obtained when an ellipsoid of the family is tangent to one of the wedge boundaries. Thus the constrained solution is one of the two for which

$$\dot{Y}_T = K_1 \dot{X}_T \qquad \text{or} \qquad \dot{Y}_T = K_2 \dot{X}_T \qquad [20]$$

It was noted earlier that when a precise relation, like [20], is obtained between $\dot{Y}_T$ and $\dot{X}_T$, the fundamental system [14] reverts to a system of equations in only two unknowns, and the least squares solution can be obtained using a simplified generalized inverse.

Thus in the case of (wedge) direction constraint, the fundamental system [14] can be solved twice, once with $\dot{Y}_T = K_1 \dot{X}_T$, and once again with $\dot{Y}_T = K_2 \dot{X}_T$; the best of the two solutions (least squares sense) is the solution of the constrained direction minimization problem. Of course, if the direction constraint wedge is a plane, only one such solution results.

3.4 Constrained Direction and Velocity

The most realistic constraint case is perhaps one in which the source direction is known to lie between two intersecting half-lines and the source speed between two values $V_1 < V_2$. In this case, the constraint region is a cylinder parallel to the f-axis, as in Figs. 2 and 3, but having for its base, in the $\dot{X}_T$-$\dot{Y}_T$ plane, a figure bounded by four arcs which are the intersections of an annular ring with an angular sector, measured from the center of the ring.

Again, if the unconstrained solution lies within the constraint region, the problem is solved. If it does not, the best constrained solution will then lie on the boundary of the constraint region, at the point where the smallest ellipsoid described by equation [17] becomes tangent to the constraint region, by increasing S_1.

Although finding the point(s) of tangency in this case requires more care than it did in the simple circular case of speed constraint, it is not difficult to develop an algorithm which will do so. The fact that the ellipsoids of the family described by [17] never change their center or their principal axes as S_1 is increased helps.

4.0 ESTIMATION OF RANGE

4.1 Rough Range Estimation

The first method discussed depends upon certain approximations to the range vector in the fundamental equation for apparent frequency

$$f_a \, (C + \vec{r} \cdot \vec{V_T}) = f \, (C + \vec{r} \cdot \vec{V_S}) \qquad [21]$$

The approximations will be better for short tracking intervals and long ranges. The principal limitation of the rough range estimate is error in frequency estimation, f. With an external input of frequency, the method could be quite accurate but, if frequency is badly determined, the rough range estimate will be useless. Further, as discussed below, the rough range estimate is unreliable when both source and searcher travel along or near the line which joins them.

Equation [21] can be rewritten as

$$C \, \frac{f_a - f}{f} = \vec{r} \cdot (\vec{V_S} - \frac{f_a}{f} \vec{V_T})$$

and f_a/f may, quite acceptably, be approximated by 1, leading to:

$$C \, \frac{f_a - f}{f} = \vec{r} \cdot (\vec{V_S} - \vec{V_T}) \qquad [22]$$

To simplify the discussion, we will assume that $\vec{V}_S$ is constant over the measurement period or, equivalently, that a subset of the measurements taken when $\vec{V}_S$ is constant is used for the estimate. A slightly more elaborate procedure can be developed in which $\vec{V}_S$ can vary and all observations can be used.

Summing [22] over the N usable observations yields

$$C \sum_{i=1}^{N} \frac{f_{a_i} - f}{f} = (\vec{V}_S - \vec{V}_T) \cdot \sum_{i=1}^{N} \vec{r}_i \qquad [23]$$

This summation reduces the importance of independent random errors in the measurements of the f_{a_i}, but an examination of the effect of the initial range, R_1, on the right side of [23] reveals the sensitivity of this equation to determination of the frequency f.

If R_1 is large and the time interval between measurements is relatively small, there will be little change in the unit direction vectors r_i, and we can write

$$\sum_{i=1}^{N} \vec{r}_i = N \vec{r}_1 \qquad [24]$$

where $\vec{r}_1$ is the initial unit direction vector. On the other hand, if R_1 is close to 0, the sum of the unit vectors is

$$\sum_{i=1}^{N} \vec{r}_i = \vec{r}_1 + (N - 1) \frac{\vec{V}_T - \vec{V}_S}{||\vec{V}_T - \vec{V}_S||} \qquad [25]$$

for the first measurement is taken in the direction $\vec{r}_1$ of the source, but all subsequent ones are in the direction of the relative velocity $\vec{V}_T - \vec{V}_S$. (More complex expressions are clearly needed here, and below, if $\vec{V}_S$ is allowed to vary). The expressions [24] and [25] can be used to bound values on the right side of [23], as follows:

$$\vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T) - (N-1) \, ||\vec{V}_S - \vec{V}_T|| \leq (\vec{V}_S - \vec{V}_T) \cdot \sum_{i=1}^{N} \vec{r}_i \qquad [26]$$
$$\leq N \vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T)$$

where $||\vec{V}_S - \vec{V}_T||$ is the magnitude, or length of the vector $\vec{V}_S - \vec{V}_T$, and is positive. It is clear that the lower bound in [26] is always inferior to the upper one, although both may be negative. The upper bound is negative only when the (signed) projection of the vector $\vec{V}_S - \vec{V}_T$ onto the direction $\vec{r}_1$ is negative, or when the (signed) projection of $\vec{V}_T - \vec{V}_S$ onto $\vec{r}_1$ is positive, or when the source is moving away from the searcher.

Since [23] can be used to replace the middle expression in the inequality [26], the following inequality results:

$$\vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T) - (N-1) \, ||\vec{V}_S - \vec{V}_T|| \leq C \sum_{i=1}^{N} \frac{f_{a_i} - f}{f} \qquad [27]$$
$$\leq N \vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T)$$

It is in the form [27] that the sensitivity of the rough range estimate (still to be described) is most evident. If, for example $\vec{V}_T - \vec{V}_S$ lies close to the line of action of $\vec{r}_1$, then $\vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T)$ is negative,

and close in value to $||\vec{V}_S - \vec{V}_T||$. In this case the two bounds in [27] differ by little. Then, if the estimate of $\vec{V}_T$ is trustworthy, [27] might be used to provide a worthwhile estimate of f; however, unless f is well known, [27] will not be useful in finding the value of R_1.

If the bounds in [27] are spaced far enough apart, and if a reliable estimate of f is available, a method for rough range determination results from closer examination of the quantity

$$(\vec{V}_S - \vec{V}_T) \cdot \sum_{i=1}^{N} \vec{r}_i = Q(R_1) \qquad [28]$$

The inequality [26] shows the maximum possible variation of $Q(R_1)$ as R_1 varies from 0 to ∞ and, if the measured values f_{a_i} satisfy the inequality [27], a value of R_1 can be found to make [23] true. Such a value of R_1 would be the rough range estimate.

To examine the behavior of $Q(R_1)$ as R_1 is varied, two other approximations are made to the unit vectors $\vec{r}_1$. If TT_i represents the time at which measurement i is made after the first measurement, i.e. $TT_1 = 0$, then

$$\vec{r}_i = \frac{\vec{R}_1 + TT_i \ (\vec{V}_T - \vec{V}_S)}{||\vec{R}_1 + TT_i \ (\vec{V}_T - \vec{V}_S)||} \qquad [29]$$

It should be noted here that the time TT_i is not modified to obtain the position of the source at the moment of the emission of pulse i rather than at the time of its reception. Such corrections are made later in

this section as well as in computer programs to allow for the source
movement during transit time of the pulse. Here, however, all times are
measured from the time of reception of the first pulse so that only a
minimal correction would be needed for the small change of range between
the first measurement and measurement i.

Further, approximating the denominator in [29] by

$$R_1 + TT_i \, \vec{r}_1 \cdot (\vec{V}_T - \vec{V}_S)$$

which represents the value of R_1 corrected by the change in range
produced by the projection of relative velocity in the $\vec{r}_1$ direction,
results in an approximation to [29]:

$$\vec{r}_i = \frac{\vec{R}_1 + TT_i \quad (\vec{V}_T - \vec{V}_S)}{R_1 + TT_i \vec{r}_1 \cdot (\vec{V}_T - \vec{V}_S)} \tag{30}$$

Now, if [30] is substituted into [28] and if the result is
differentiated with respect to R_1, the following derivative is obtained:

$$\frac{dQ}{dR_1} = (||\vec{V}_S - \vec{V}_T||^2 - (\vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T))^2) \sum_{i=1}^{N} \frac{TT_i}{(R_1 + TT_i \vec{r}_1 \cdot (\vec{V}_T - \vec{V}_S))^2} \tag{31}$$

and the quantity is always positive.

Finally, when the inequality [27] is satisfied by the estimated f
and the measured f_{a_i}, the quantity $Q(R_1)$ rises steadily with increasing
R_1, one unambiguous value of R_1 can be found for which [23] is
satisfied. This value is the rough range estimate. The easiest way to

find it is to apply the method of Newton-Raphson to [31]. The estimates are found with the APL function RRNG of Appendix A.

One further look at the form of the derivative, in [31], reveals again the sensitivity of this estimate to the direction of $\vec{r}_1$, and the relative velocity $\vec{V}_T - \vec{V}_S$. Since the derivative contains the factor

$$(||\vec{V}_S - \vec{V}_T||^2 - (\vec{r}_1 \cdot (\vec{V}_S - \vec{V}_T))^2)$$

it may be very small when $\vec{V}_T - \vec{V}_S$ lies in the line of action of $\vec{r}_1$. The range estimate then becomes quite unreliable, and the Newton-Raphson root-finding technique may have to be abandoned for another technique which is described below.

4.2 Best Linear Range Match

The second method for possible range determination also uses the least squares solution

$$STV = <f, \dot{X}_T, \dot{Y}_T> \qquad [32]$$

of the linear system:

$$\dot{X}_T \cos B_1 + \dot{Y}_T \sin B_1 - f \frac{C + \vec{r}_1 \cdot \vec{V}_{S_1}}{f_{a_1}} = -C$$

$$[33]$$

$$\dot{X}_T \cos B_N + \dot{Y}_T \sin B_N - f \frac{C + \vec{r}_N \cdot \vec{V}_{S_N}}{f_{a_N}} = -C$$

In finding the least squares solution STV, the actual measured values of bearing B_1, .., Γ_N were used to evaluate the coefficients of system [33]. Once values of f, $\dot{X}_T$, $\dot{Y}_T$ are available, one can

a) assume a value of range at initial measurement, R_1;

b) use the measured initial bearing, the estimated velocity $\dot{X}_T$, $\dot{Y}_T$, together with the assumed value of R_1 to generate a set of bearings B_2, B_3,.., B_N at the later measurement times;

c) calculate the left sides of the equations in [33] and compute the summed, squared, differences between left and right sides in [33], using the results of b) and the other known and measured quantities C, V_{S_i}, f_{a_i}.

The result obtained in c), called DELTASQ in the APL functions of Appendix A, can be considered a measure of merit of the assumed range R_1, subject to the accuracy of the estimate STV. If there is a minimum value of DELTASQ over the interval of possible assumed ranges, R_1, the value of R_1 corresponding to the minimum could be chosen as the best linear range match.

The above computational procedure is easily implemented. The calculation is performed with the APL function RNSEL of Appendix A.

4.3 Least Squares Iteration

The third method for range estimation is the classical least squares iteration that permits to find the point in parameter space, $<f, \dot{X}_T, \dot{Y}_T, R_1>$, which minimizes a squared-summed error function.
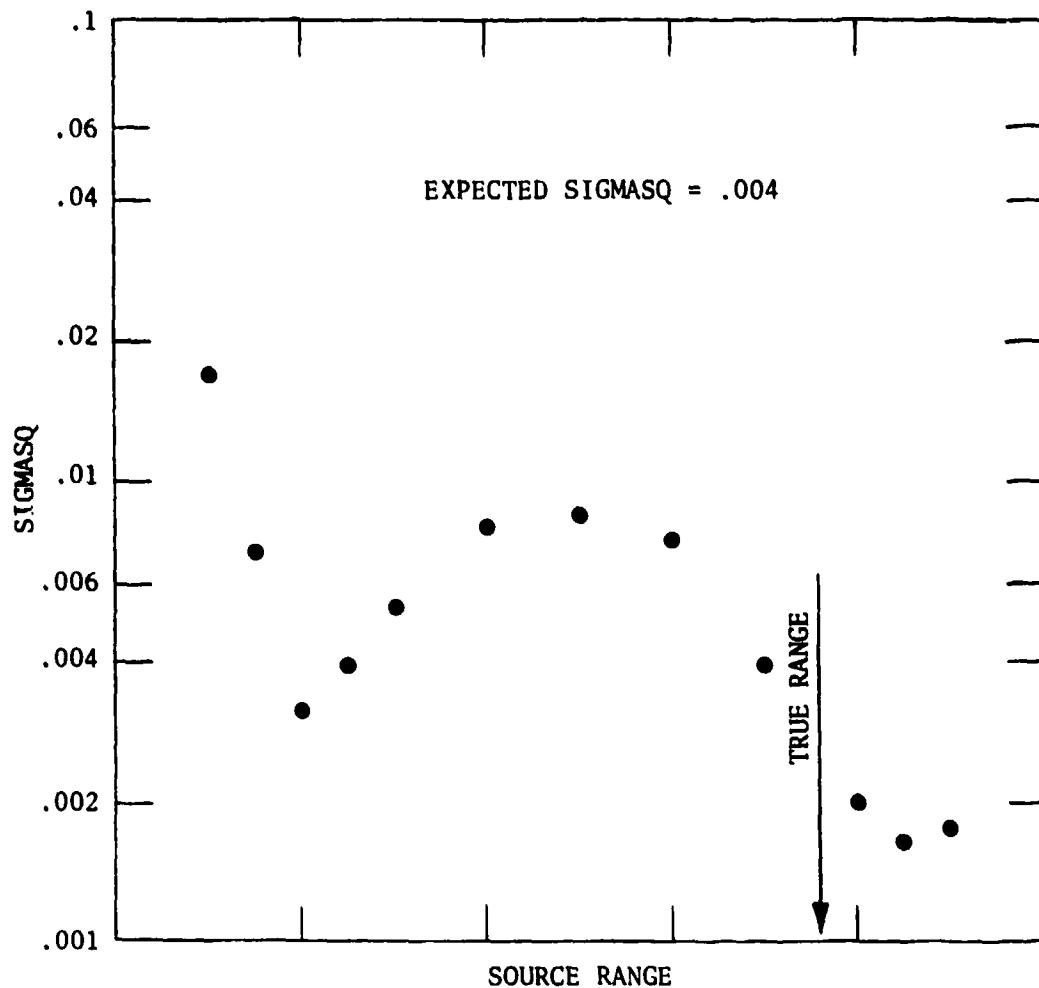
Figure 4 - Example of range estimation for two searchers

Like the best linear range method or second method previously explained, the third method uses values of the parameters to generate a set of bearings, so that its comparison with measured data involves only frequency measurements.

In the second method, only the range parameter can be varied. In the third method, the minimum is sought by evaluating the zero points of partial derivatives (of the squared-summed error function). It is thus possible to vary any of the parameters.
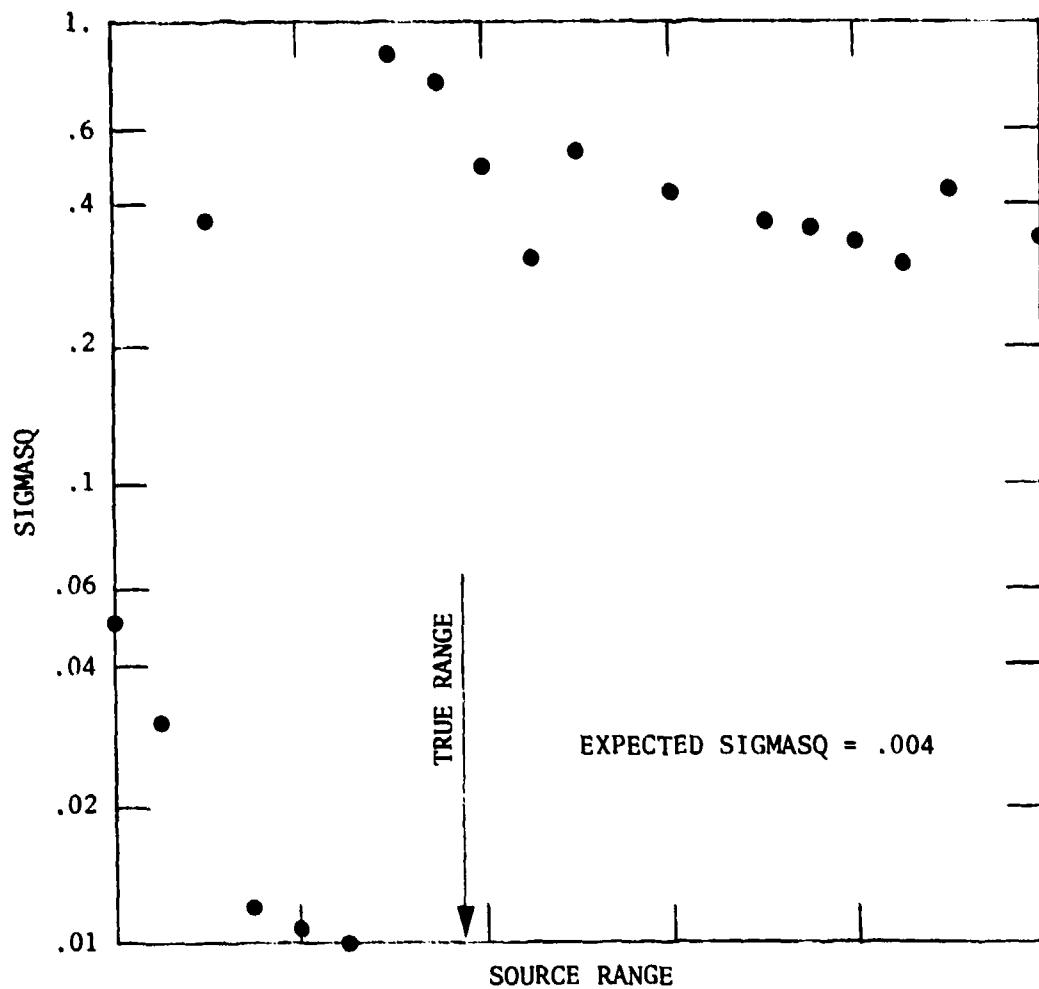
Figure 5 - Example of range estimation for one searcher

Unfortunately, in the presence of measurement noise, the squared-summed error surface for frequencies seems to possess many distinct relative minima. Computation reveals that these are not always located near the true values of the parameters but, in the search for the zeros of partial derivatives, incorrect minima often arise. Figure 4 shows the result of a typical calculation in which a search was performed on a range grid, and the algorithm was allowed to seek the best values of f, X_T, Y_T, to minimize SIGMASQ in the APL function KITERATE. In this case, two searchers are used and the range is not

excessive; so it is a favorable case. The figure shows that, without a good initial range estimation, an algorithm searching in range as well as in the other parameters might yield several different values. The behavior pictured is not exceptional.

The case presented in Fig. 5 was calculated for one searcher. Although the single searcher was allowed to change course, and the same total number of observations was used as in the two-searcher case of Fig. 4, it is apparent that many more local minima exist. Counting end point minima, there are at least five in Fig. 5, and an iteration might converge to any one.

Despite the complexity of the function to be minimized, the least squares algorithm is useful in many cases and it often yields a deepest minimum not far from the true value of the range parameter. The basic equations for the method are presented next.

Using [21], solved for the apparent frequency, f_a, the squared-summed differences for the frequency observations can be expressed as

$$\text{SIGMASQ} = \sum_{i=1}^{N} (f \, \frac{C + \vec{r_i} \cdot \vec{V_S}}{C + \vec{r_i} \cdot \vec{V_T}} - f_{a_i})^2 \qquad [34]$$

SIGMASQ is to be minimized by fixing a range, then seeking points where the partial derivatives below vanish.

$$\frac{\partial}{\partial f} = 2 \sum_{i=1}^{N} (f \frac{C + \vec{r_i} \cdot \vec{V_S}}{C + \vec{r_i} \cdot \vec{V_T}} - f_{a_i}) \frac{C + \vec{r_i} \cdot \vec{V_S}}{C + \vec{r_i} \cdot \vec{V_T}}$$

$$\frac{\partial}{\partial \dot{X_T}} = 2 \sum_{i=1}^{N} (\qquad) \frac{\partial f_a}{\partial \dot{X_T}}$$

[35]

$$\frac{\partial}{\partial \dot{Y_T}} = 2 \sum_{i=1}^{N} (\qquad) \frac{\partial f_a}{\partial \dot{Y_T}}$$

where the f_a, on the right, is the function of $\dot{X_T}$ and $\dot{Y_T}$ obtained by solving [21] for f_a. However, these formulas must be interpreted carefully. For each i, $1 \le i \le N$, $f_a(\dot{X_T}, \dot{Y_T})$ is a different function and, hence, each equation of [35] is a scaler product between data vectors, of length N. It is useful to recognize this fact in notation, by using a scaler product notation between data vectors of length N, and by emphasizing such vectors by the notation ⟵⟶. Then, if the error vector whose components appear in the parenthesis on the right sides in [35] is denoted $\overleftrightarrow{EV}$, [35] can be rewritten in compact form as follows:

$$\frac{\partial}{\partial f} = 2 \, EV \, (\frac{C + \vec{r_i} \cdot \vec{V_S}}{C + \vec{r_i} \cdot \vec{V_T}})$$

$$\frac{\partial}{\partial \dot{X_T}} = 2 \, \overleftrightarrow{EV} \odot \frac{\overleftrightarrow{\partial f_a}}{\partial \dot{X_T}}$$

[35']

$$\frac{\partial}{\partial \dot{Y_r}} = 2 \, \overleftrightarrow{EV} \odot \frac{\overleftrightarrow{\partial f_a}}{\partial \dot{Y_T}}$$

Then, $\dfrac{\partial \overleftrightarrow{f}_a}{\partial \vec{X}_T}$ is an N-vector whose ith component is:

$$\frac{f}{(C+\vec{r}_i \cdot \vec{V}_T)^2}((C+\vec{r}_i \cdot \vec{V}_T)(\vec{V}_{S_i} \cdot \frac{\partial \vec{r}_i}{\partial \vec{X}_T})-(C+\vec{r}_i \cdot \vec{V}_{S_i})(\cos B_i+\vec{V}_T \cdot \frac{\partial \vec{r}_i}{\partial \vec{X}_T})) \qquad [36]$$

and there is a similar expression for the ith component of $\dfrac{\partial \overleftrightarrow{f}_a}{\partial \vec{Y}_T}$.

Thus, the evaluation of the needed partial derivatives cannot be completed without evaluating the expressions

$$\frac{\partial \vec{r}_i}{\partial \vec{X}_T} \qquad \text{and} \qquad \frac{\partial \vec{r}_i}{\partial \vec{Y}_T} \; .$$

It must now be recalled that $\vec{r}_i$ is the unit vector pointing from the position of the searcher at time i, TT_i, toward the position of the source at a time TTR_i at which the ith pulse was emitted. In other words, the source position times must be corrected for transmission time. This correction is performed with the APL function NUVECT, for use in the iteration function KITERATE.

In terms of TT_i and TTR_i, $\vec{r}_i$ can be written as:

$$\vec{r}_i = \frac{\vec{R}_i}{R_i} = \frac{1}{R_i} \; <R_1 \cos B_1 + \dot{X}_T \, TTR_i - \dot{X}_S \, TT_i, \qquad\qquad [37]$$
$$R_1 \sin B_1 + \dot{Y}_T \, TTR_i - \dot{Y}_S \, TT_i>$$

where R_i is the magnitude of the vector in corners <>.

Next, [37] is differentiated with respect to $\dot{X}_T$ and $\dot{Y}_T$, to yield the two needed derivatives. Note that no approximation has been made in obtaining these derivatives:

$$\frac{\partial r_i}{\partial \dot{X}_T} = \frac{TTR_i}{RRN^3} <RRN^2 - RRX^2, - RRX \times RRY>$$

[38]

$$\frac{\partial r_i}{\partial \dot{Y}_T} = \frac{TTR_i}{RRN^3} <-RRX \times RRY, RRN^2 - RRY^2>$$

where

$$RRX = R_1 \cos B_1 + TTR_i \times \dot{X}_T - TT_i \times \dot{X}_S$$

$$RRY = R_1 \sin B_1 + TTR_i \times \dot{Y}_T - TT_i \times \dot{Y}_S$$

[39]

$$RRN = (RRX^2 + RRY^2)^{\frac{1}{2}}$$

and these quantities represent the X and Y coordinates of the source as well as distance in a corrected searcher relative space in which the source position at the time of emission of pulse i is related to the searcher position at the time of reception of pulse i.

The expressions [38] substituted into [36], substituted into [35'] yield formulas for the needed partial derivatives. It is pointless to write out these complete formulas here. The required quantities are computed in the APL function FCN.

Finally, the search for the zero points of the derivatives of [35'] is carried out by a Newton-Raphson procedure for which the required derivative is calculated numerically with the APL function JACOB, and the actual iteration to the convergence criterion is performed with KITERATE. The search is managed by a control function named SSSTM.

5.0 USE OF ALGORITHMS

Computer programs are available for only two of the algorithms analyzed above. DOPPVEC2 estimates f, $\dot{X}_T$ and $\dot{Y}_T$ with speed constraint. SSSTM estimates target range by least squares iteration from the DOPPVEC2 results and then makes a new estimation of f, $\dot{X}_T$ and $\dot{Y}_T$. These computer programs are listed in Apprendix A.

If the reader uses these programs, he may notice a few peculiarities of behavior which might usefully be pointed out here:

1) If bearing and frequency measurements are totally free of error, DOPPVEC2 will yield an exact estimation of f, $\dot{X}_T$, $\dot{Y}_T$. SSSTM, however, will yield estimates with small errors because the search in range is carried out at discrete values which may not correspond to true range.

2) The variable SIGMASQ does not always pass through minima; it may decrease indefinitely as range increases in the interval searched by SSSTM. In such cases, the estimated value of range is always near the high end of the interval and may bear no resemblance at all to the true value.

3) In principle, SSSTM improves the estimate of f, $\dot{X}_T$, and $\dot{Y}_T$ made by DOPPVEC2. In reality, the new estimate may be worse than the original.

4) Both algorithms favor speed estimates close to the speed constraint rather than to the real speed. In most cases, the constrained solution is a bounding solution in the constraint region.

6.0 CONCLUSION

The operational effectiveness of the algorithms discussed above is unknown. It can be estimated by simulation studies in which searcher tactics are taken into account. Such studies have now been undertaken for the two algorithms already programmed. The other algorithms analyzed above should be easy to program whenever needed.

7.0 <u>REFERENCES</u>

1. Ben-Israel and Greville, "Generalized Inverses, Theory and
    Applications", New York, Wiley-Interscience, 1974.

APPENDIX A

THE APL FUNCTIONS

The following APL functions are presented, and briefly described.

1) MEASGEN: Accepts the basic descriptions of source and searcher(s), and generates bearing and frequency inputs, with random errors, for use by the processing algorithms.

2) DOPPVEC2: Performs the least-squares solution of the fundamental system of linear equations, subject to a speed constraint.

3) NUVECT: Uses the measurement times and the first measured bearing to convert an estimate, STV, of f, $\dot{X}_T$, $\dot{Y}_T$, and an estimate of range, RN, into predicted source coordinates and frequencies, for comparison with the measurements generated by MEASGEN.

4) RMNL: Chooses a vector of N random numbers from a Gaussian distribution of mean P[1] and standard deviation P[2]. The choices are independent.

5) SIGM: Computes the summed-squared deviation for frequency data, SIGMASQ.

6) FCN: Computes the values of the partial derivatives of SIGMASQ, with respect to the parameters f, $\dot{X}_T$, $\dot{Y}_T$.

7) JACOB: Performs a numerical calculation of the matrix of second partial derivatives of SIGMASQ, for use in the iteration KITERATE.

8) KITERATE: Does the Newton-Raphson iteration to find a zero-point for the partial derivatives of SIGMASQ.

9) RRNG: Computes the rough range estimate.

10) RNSEL: Chooses the best linear range estimate for the solution of the fundamental linear system.

11) SSSTM: Performs a grid search, using KITERATE, for a range at which the smallest value of SIGMASQ can be obtained, by simultaneously varying the other parameters.

12) DAPPROX, DEXACT, SHIFT and SHIFTY: Solve the exact and approximate formulas for the Doppler shift, as presented in Section 2.0.

Further comments accompany certain of the individual APL functions.

Function 1: MEASGEN

INPUTS: TAR is a 1 x 8 vector which consists of the following:

TAR[1] Source initial X

TAR[2] Source initial Y

TAR[3] Source X velocity

TAR[4] Source Y velocity

TAR[5] Maximum possible source speed

TAR[6] Default estimate of source range (to be used if all
        efforts to find range fail)

TAR[7] Source emission frequency

TAR[8] Speed of sound in the medium

Since only one speed vector is given for it, the source moves on
a straight line, at least throughout the measurement interval.

SRCH1 is a 1 x 7 vector which consists of :

SRCH1[1] Searcher initial X

SRCH1[2] Searcher initial Y

SRCH1[3] Searcher initial X velocity

SRCH1[4] Searcher initial Y velocity

SRCH1[5] Searcher turn time

SRCH1[6] Searcher final X velocity

SRCH1[7] Searcher final Y velocity

SRCH2 is a 1 x 7 vector with the same components as SRCH1,
describing a second searcher. Although only two searchers are provided
for and only a simple dogleg path is given for each one there are no
limits in the algorithms which would prevent reprogramming for
additional searchers, and more complicated search paths. Searchers can
be prevented from maneuvering by setting SRCH1 [5] and SRCH2 [5] longer
than the last measurement time in TIMEM, and the present programs can be
used for either 1 or 2 searchers, as controlled by the input TIMEM,
detailed below.

MEASM is a 2 x 5 matrix of measurement parameters, as follows:

MEASM[1;1]	1
MEASM[2;1]	2
MEASM[1;2]	The mean of the bearing error distribution
MEASM[1;3]	The standard deviation of bearing error
MEASM[1;4]	The mean of the frequency error distribution
MEASM[1;5]	The standard deviation of frequency error
MEASM[2;2]-[2;5]	Same as MEASM [1;2] - [1;5] for the second searcher; these parameters can, of course, be given different values for each searcher.

TIMEM is an N x 2 matrix which specifies the times of measurements. The first column specifies the number of the searcher that makes the measurement, and the second one gives the times. While the present algorithms do not consider loss of contact, this factor could be introduced by producing a random generator for contact times and searcher numbers, or by accepting TIMEM as an output from a search algorithm.

To use MEASGEN for a single searcher, the first column can be set entirely to 1 (or to 2), or else vectors SRCH1 and SRCH2 can be made identical.

FUNCTION: Lines [1] through [11] set the measurement times, together with the searcher coordinates and velocities at those times. True measurement times are used. Lines [12]-[15] find source relative coordinates in searcher relative space(s) at measurement times. Lines [16]-[25] perform corrections to obtain the source positions at the time of pulse emissions, relative to searcher positions at times of

reception. The remaining lines calculate true bearings and frequencies and lines [35], [36] apply normally distributed independent errors.

OUTPUT: The output of MEASGEN is an N x 12 matrix called DATAM. Each line stores the information appropriate to a measurement, with components as follows:

DATAM[I;1]   Line number I

DATAM[I;2]   Searcher number for line I

DATAM[I;3]   Measurement time

DATAM[I;4]   Bearing

DATAM[I;5]   Frequency

DATAM[I;6]   Searcher X coordinate at measurement time

DATAM[I;7]   Searcher Y coordinate at measurement time

DATAM[I;8]   Searcher X velocity, this time

DATAM[I;9]   Searcher Y velocity, this time

DATAM[I;10]   Range from searcher 1 to source, this time

DATAM[I;11]   Range from searcher 2 to source, this time

DATAM[I;12]   Turn time of the searcher on this line

Function 2: DOPPVEC2

INPUT: DATAM, from MEASGEN

FUNCTION: Lines [1] - [10] compute the matrix, M, of the fundamental linear system and obtain the least squares solution, using the Moore-Penrose inverse M̄ programmed into APL. Line [16] and [17] check that the speed constraint specified in TAR [8] is met. If not,

lines [18] through [32] set the geometrical parameters of the ellipses described in the text, and the remainder of the function uses Newton-Raphson to solve for the smallest value of DELTASQ which satisfies the speed constraint.

OUTPUT: STV is the least squares solution of the linear system, subject to the speed constraint.

Function 3: NUVECT

INPUTS: DATAM, from MEASGEN

STV, from DOPPVEC2, or from KITERATE when used for iteration

RN, from SSTM, which controls the grid search for range.

FUNCTION: Lines [1] through [12] use the searcher information from each line in DATAM to generate relative source coordinates and distance, in searcher relative space, at the time of reception of a pulse. Lines [13] and [14] correct the source position to the emission time of the pulse. After position correction, the remaining lines compute the angular and frequency data to be expected. The entire operation is performed simultaneously for the whole of the matrix DATAM.

OUTPUT: The relative position and distance vectors (RRX, RRY, RRN), the bearing data (COSV, SINV), and the components of the Doppler formula S, CRVT) are all used in other functions. Vectors NUAV, the predicted observed frequencies, and TTVR, the corrected emission times, are perhaps the principal outputs of NUVECT.

Function 6:   FCN

> INPUTS: DATAM plus the position and frequency predictions
> of NUVECT.

> FUNCTION: Lines [3], [7], [8] compute the three partial
derivatives of SIGMASQ, with respect to f, $\dot{X}_T$, $\dot{Y}_T$, respectively.

> OUTPUT: FV is a 3 x 1 matrix of the values of the partial
> derivatives.

Function 7:   JACOB

> INPUTS: FV, or FCN STV, from FCN

> FUNCTION: Computes numerical approximations for the nine second
partial derivatives of the three first partials in FV, with respect to
each of the three parameters f, $\dot{X}_T$, $\dot{Y}_T$.   Instead of a true derivative, a
difference quotient is computed for an increment in f of 10^{-4} and
increments in $\dot{X}_T$, $\dot{Y}_T$ of 10^{-6}.

> OUTPUT: The 3 x 3 matrix JA, or JACOB STV.

Function 8:   KITERATE

> INPUTS: DATAM, TAR, FCN STV, JACOB STV, REPIT,
> outputs of NUVECT.

FUNCTION: Lines [4] and [5] compute lower and upper frequency bounds (LNUB, UNUB) for the frequency parameter, based upon the observed frequencies and the speed constraint.

KITERATE attempts to move from point STV, in a direction indicated by JACOB STV, to find another parameter vector, STVA, which satisfies the speed and frequency constraints, and results in a decreased value of SIGM STV. The base values are referred to as SIGMB, FCNB, STVB. If the new value STVA of line [11] does not satisfy the restrictions in [13], other values of STVA are tried by progressive halving of the parameter vector displacement, [13], until one is found that satisfies the restrictions; if it is impossible, then we revert to the previous estimate, line [17].

If improved values of STV are found, this process continues until the difference of summed squared deviations, SIGM STV, stabilizes to within $\pm 10^{-5}$ until a predetermined number of repetitions, REPIT, is reached.

OUTPUTS: STV, STVA, both names for the same parameter vector.

Function 11: SSSTM.

INPUTS: STV, from DOPPVEC2.

FUNCTION: Lines [1] through [7] calculate KITERATE STV for the initial input STV, with values of RN from 10 to 120. The results are stored in a matrix SSTM, as indicated in line [5]. Lines [8] through [15] construct a vector, AV, of ones and zeroes with a one in each

position where the corresponding row of SSTM exhibits a relative minimum value of SIGMA.

If there are no relative minima in column 5 of SSTM, line [17] prints a row of zeroes, as row [13] of SSTM, and terminates.

Line [19] selects the relative minimum rows of SSTM and places them in a matrix called RNS. Lines [20] through [32] expand the search about the relative minima of SSTM in increments of -5, 0, +5 about each relative minimum.

These calculations use the values of STV found by KITERATE at the relative minimum ranges, not the original STV from DOPPVEC2. The results of the expanded calculations are recorded as new rows in the matrix SSTM.

Line [33] selects the absolute minimum in column 5 of the expanded matrix SSTM. Finally, lines [34] through [45] further refine the search in increments of -2.5, 0, +2.5 about the range value found by [33], and records the output MIN.

OUTPUT: MIN.

Function 12: DAPPROX, DEXACT, SHIFT, SHIFTY.

DAPPROX computes the DOPPLER approximation of formula [7], Section 2.0, as DAA, line [2]. Line [3] computes formula [9'], as DAA2. DEXACT solves the exact relation [5] for Δ_a, by a Newton-Raphson iteration, to accuracy $10^{-10}\Delta_a$. The functions SHIFT and SHIFTY simply

permit the values of exact and approximate solutions to be computed and displayed for a number of different frequencies, and for a number of different velocities of source and searcher.

The vector input IV consists of:

Source $\dot{X}$, Source $\dot{Y}$, Searcher $\dot{X}$, Searcher $\dot{Y}$

N is the desired number of repetitions

D is the interpulse interval, D = 1/f.

SHIFT makes N random choices of the relative coordinates X and Y, while SHIFTY demands initial $\dot{X}$ and $\dot{Y}$, as IV[5], IV[6], and moves the source along a straight line until N positions have been achieved.

```
        ∇MEASGEN[□]∇
        ∇MEASGEN;TV;L;S1XV;S1YV;S2XV;S2YV;RELX1;RELY1;RELX2;RE
        LY2;BEAR2;RV1;RV2;FREQ2;N;S
[1]     TV←QTIMEM[;2]
[2]     TPX←TAR[1]+TAR[3]×TV
[3]     TPY←TAR[2]+TAR[4]×TV
[4]     S1XP←((SRCH1[1]+SRCH1[3]×TV)×(TV≤SRCH1[5]))+((SRCH1[1]
        +SRCH1[3]×SRCH1[5])+(TV-SRCH1[5])×SRCH1[6])×(TV>SRCH1[
        5])
[5]     S1YP←((SRCH1[2]+SRCH1[4]×TV)×(TV≤SRCH1[5]))+((SRCH1[2]
        +SRCH1[4]×SRCH1[5])+(TV-SRCH1[5])×SRCH1[7])×(TV>SRCH1[
        5])
[6]     S2XP←((SRCH2[1]+SRCH2[3]×TV)×(TV≤SRCH2[5]))+((SRCH2[1]
        +SRCH2[3]×SRCH2[5])+(TV-SRCH2[5])×SRCH2[6])×(TV>SRCH2[
        5])
[7]     S2YP←((SRCH2[2]+SRCH2[4]×TV)×(TV≤SRCH2[5]))+((SRCH2[2]
        +SRCH2[4]×SRCH2[5])+(TV-SRCH2[5])×SRCH2[7])×(TV>SRCH2[
        5])
[8]     S1XV←(SRCH1[3]×(TV≤SRCH1[5]))+SRCH1[6]×(TV>SRCH1[5])
[9]     S1YV←(SRCH1[4]×(TV≤SRCH1[5]))+SRCH1[7]×(TV>SRCH1[5])
[10]    S2XV←(SRCH2[3]×(TV≤SRCH2[5]))+SRCH2[6]×(TV>SRCH2[5])
[11]    S2YV←(SRCH2[4]×(TV≤SRCH2[5]))+SRCH2[7]×(TV>SRCH2[5])
[12]    RELX1←TPX-S1XP
[13]    RELY1←TPY-S1YP
[14]    RELX2←TPX-S2XP
[15]    RELY2←TPY-S2YP
[16]    RV1←((RELX1*2)+RELY1*2)*.5
[17]    RV2←((RELX2*2)+RELY2*2)*.5
[18]    TPX1←TPX-TAR[3]×RV1÷TAR[8]
[19]    TPX2←TPX-TAR[3]×RV2÷TAR[8]
[20]    TPY1←TPY-TAR[4]×RV1÷TAR[8]
[21]    TPY2←TPY-TAR[4]×RV2÷TAR[8]
[22]    RELX1←TPX1-S1XP
[23]    RELY1←TPY1-S1YP
[24]    RELX2←TPX2-S2XP
[25]    RELY2←TPY2-S2YP
[26]    BEAR1←((¯3○RELY1÷RELX1)×(RELX1>0)×(RELY1≥0))+(((¯3○REL
        Y1÷RELX1)+○1)×(RELX1<0))+(((¯3○RELY1÷RELX1)+○2)×(RELX1
        >0)×(RELY1<0))
[27]    BEAR1←BEAR1+((○1.5)×(RELX1=0)×(RELY1<0))+(.5)×(RELX1=0
        )×(RELY1>0)
[28]    BEAR2←((¯3○RELY2÷RELX2)×(RELX2>0)×(RELY2≥0))+(((¯3○REL
        Y2÷RELX2)+○1)×(RELX2<0))+(((¯3○RELY2÷RELX2)+○2)×(RELX2
        >0)×(RELY2<0))
[29]    BEAR2←BEAR2+((○1.5)×(RELX2=0)×(RELY2<0))+(.5)×(RELX2=0
        )×(RELY2>0)
[30]    RV1←((RELX1*2)+RELY1*2)*.5
[31]    RV2←((RELX2*2)+RELY2*2)*.5
[32]    FREQ1←TAR[7]×((RV1×TAR[8])+(RELX1×S1XV)+RELY1×S1YV)÷(R
        V1×TAR[8])+(RELX1×TAR[3])+RELY1×TAR[4]
```

```
[33]    FREQ2←TAR[7]×((RV2×TAR[8])+(RELX2×S2XV)+RELY2×S2YV)+(R
        V2×TAR[8])+(RELX2×TAR[3])+RELY2×TAR[4]
[34]    L←(ρTIMEM)[1]
[35]    BEAR←(2,L)ρ(BEAR1+L RMNL 1+(¯2+MEASM[1;])),BEAR2+L RMN
        L(1+(¯2+MEASM[2;])))
[36]    FREQ←(2,L)ρ(FREQ1+L RMNL(3+MEASM[1;])),FREQ2+L RMNL(3+
        MEASM[2;])
[37]    SERX←(2,L)ρ,S1XP,S2XP
[38]    SERY←(2,L)ρ,S1YP,S2YP
[39]    RELVX←(2,L)ρ,S1XV,S2XV
[40]    RELVY←(2,L)ρ,S1YV,S2YV
[41]    DATAM←1 12ρ0
[42]    N←1
[43]    S←TIMEM[1;1]
[44]    DATAM[1;]←1,S,TIMEM[1;2],BEAR[S;1],FREQ[S;1],SERX[S;1]
        ,SERY[S;1],RELVX[S;1],RELVY[S;1],RV1[1],RV2[1],((S=1)×
        SRCH1[5])+(S=2)×SRCH2[5]
[45]    N←N+1
[46]    →(N>L)/0
[47]    S←TIMEM[N;1]
[48]    DATAM←DATAM,[1]N,S,TIMEM[N;2],BEAR[S;N],FREQ[S;N],SERX
        [S;N],SERY[S;N],RELVX[S;N],RELVY[S;N],RV1[N],RV2[N],((
        S=1)×SRCH1[5])+(S=2)×SRCH2[5]
[49]    →45
        ∇
```

```
         ∇DOPPVEC2[□]∇
         ∇DOPPVEC2;M1;M2;M3;R;M;DOPPV;VAX;VAY;NUD;W;TAM;TBN;FAM
         ;FBM;TDNUBM;TDNUAM;PSI;PSID;A11;A22;A33;A1;A2;A3;A12;A
         13;A23;B11;B22;B1;B2;B12;F;REP
[1]      LOOP←1
[2]      M1←2○DATAM[;4]
[3]      M2←1○DATAM[;4]
[4]      M3←-(+DATAM[;5])×TAR[8]+(M1×DATAM[;8])+M2×DATAM[;9]
[5]      R←(ρDATAM)[1]
[6]      M←(R,3)ρ0
[7]      M[;1]←M1
[8]      M[;2]←M2
[9]      M[;3]←M3
[10]     DOPPV←-TAR[8]×(⌹M)+.×((R,1)ρ1)
[11]     DELTASQ←+/,((M+.×DOPPV)+TAR[8])*2
[12]     VAX←DOPPV[1;1]
[13]     VAY←DOPPV[2;1]
[14]     NUD←DOPPV[3;1]
[15]     STV←NUD,VAX,VAY
[16]     W←(DOPPV[1;1]*2)+DOPPV[2;1]*2
[17]     →(W≤TAR[5]*2)/C1
[18]     A11←+/M1*2
[19]     A22←+/M2*2
[20]     A33←+/M3*2
[21]     A12←+/M1×M2
[22]     A13←+/M1×-M3
[23]     A23←+/M2×-M3
[24]     A1←+/M1
[25]     A2←+/M2
[26]     A3←+/-M3
[27]     B11←(A13*2)-A11×A33
[28]     B22←(A23*2)-A22×A33
[29]     B1←TAR[8]×(A13×A3)-A33×A1
[30]     B2←TAR[8]×(A23×A3)-A33×A2
[31]     B12←(A13×A23)-A12×A33
[32]     F←(A33×DELTASQ)+(TAR[8]*2)×(A3*2)-R×A33
[33]     PSI←3 1ρ0
[34] PS:PSI[1;1]←(B11×VAX*2)+(B22×VAY*2)+(2×B12×VAX×VAY)+(2×
         B1×VAX)
[35]     PSI[1;1]←PSI[1;1]+(2×B2×VAY)+F
[36]     PSI[2;1]←(VAX*2)+(VAY*2)-TAR[5]*2
[37]     PSI[3;1]←(B12×VAX*2)+((B22-B11)×VAX×VAY)+(B2×VAX)-(B1×
         VAY)+B12×VAY*2
[38]     PSID←3 3ρ0
[39]     PSID[1;1]←(2×B11×VAX)+(2×B12×VAY)+2×B1
[40]     PSID[1;2]←(2×B22×VAY)+(2×B12×VAX)+2×B2
[41]     PSID[1;3]←A33
[42]     PSID[2;3]←PSID[3;3]←0
[43]     PSID[2;1]←2×VAX
[44]     PSID[2;2]←2×VAY
```

```
[45]    PSID[3;1]+(2×B12×VAX)+((B22-B11)×VAY)+B2
[46]    PSID[3;2]+((B22-B11)×VAX)-(2×B12×VAY)+B1
[47]    +(LOOP>1)/R2
[48]    REP+(3 1ρ(VAX,VAY,DELTASQ))-(⊞PSID)+.×PSI
[49]    +R3
[50] R2:REP1+REP-(⊞PSID)+.×PSI
[51]    +((+/.|REP1-REP)<.03)/QS
[52]    REP+REP1
[53] R3:VAX+REP[1;1]
[54]    VAY+REP[2;1]
[55]    F+(A33×REP[3;1])+(TAR[8]*2)×(A3*2)-R×A33
[56]    LOOP+LOOP+1
[57]    +PS
[58] QS:VAX+REP1[1;1]
[59]    VAY+REP1[2;1]
[60]    NUD+((A13×VAX)+(A23×VAY)+TAR[8]×A3)÷A33
[61] C1:STV+NUD,VAX,VAY
        ∇
```

```
        ∇RMNL[⎕]∇
        ∇Z+N RMNL P;AA;Y
[1]     Y+((?Nρ100000)÷100001)+.0000001
[2]     AA+((((1-Y)*-1÷6.158)-1)*1÷4.874)
[3]     AA+(AA-(((Y*-1÷6.185)-1)*1÷4.874))÷.323968
[4]     Z+P[1]+AA×P[2]
        ∇
```

```
        ∇SIGM[⎕]∇
        ∇SIGMASQ+SIGM STV
[1]     NUVECT STV
[2]     SIGMASQ++/(NUAV-DATAM[;5])*2
        ∇
```

```
        ∇NUVECT[□]∇
        ∇NUVECT STV;TB;BB;TTV;PTXV;PTYV
[1]     CVB←DATAM[;3]<DATAM[;12]
[2]     CVA←DATAM[;3]≥DATAM[;12]
[3]     TB←DATAM[1;3]
[4]     BB←DATAM[1;4]
[5]     TTV←DATAM[;3]-TB
[6]     PTXV←((DATAM[;2]=1)×SRCH1[3])+(DATAM[;2]=2)×SRCH2[3]
[7]     PTYV←((DATAM[;2]=1)×SRCH1[4])+(DATAM[;2]=2)×SRCH2[4]
[8]     RRX←((TTV×STV[2]-DATAM[;8])×CVB)+(((DATAM[;12]-TB)×STV
        [2]-PTXV)+(TTV+TB-DATAM[;12])×STV[2]-DATAM[;8])×CVA
[9]     RRX←RRX+((DATAM[;2]=1)×(RN×2○BB))+(DATAM[;2]=2)×(RN×2○
        BB)+(SRCH1[1]+TB×SRCH1[3])-(SRCH2[1]+TB×SRCH2[4])
[10]    RRY←((TTV×STV[3]-DATAM[;9])×CVB)+(((DATAM[;12]-TB)×STV
        [3]-PTYV)+(TTV+TB-DATAM[;12])×STV[3]-DATAM[;9])×CVA
[11]    RRY←RRY+((DATAM[;2]=1)×(RN×1○BB))+(DATAM[;2]=2)×(RN×1○
        BB)+(SRCH1[2]+TB×SRCH1[4])-(SRCH2[2]+TB×SRCH2[4])
[12]    RRV←((RRX*2)+RRY*2)*.5
[13]    RRX←RRX-(RRV-RRV[1])×STV[2]÷TAR[8]
[14]    RRY←RRY-(RRV-RRV[1])×STV[3]÷TAR[8]
[15]    RRN←((RRX*2)+RRY*2)*.5
[16]    COSV←RRX÷RRN
[17]    SINV←RRY÷RRN
[18]    CRVS←TAR[8]+(((COSV×DATAM[;8])+SINV×DATAM[;9])×CVB)+((
        COSV×DATAM[;8])+SINV×DATAM[;9])×CVA
[19]    CRVT←TAR[8]+(COSV×STV[2])+SINV×STV[3]
[20]    NUAV←STV[1]×CRVS÷CRVT
[21]    TTVR←TTV-RRN÷TAR[8]
        ∇
```

```
        ∇FCN[□]∇
        ∇FV←FCN STV;EV;F1;F2;F3;PRXTD1;PRXTD2;PRYTD1;PRYTD2
[1]     NUVECT STV
[2]     EV←NUAV-DATAM[;5]
[3]     F1←2×+/EV×CRVS÷CRVT
[4]     PRXTD1←(TTVR÷RRN*3)×(RRN*2)-RRX*2
[5]     PRYTD1←PRXTD2←(TTVR÷RRN*3)×-RRX×RRY
[6]     PRYTD2←(TTVR÷RRN*3)×(RRN*2)-RRY*2
[7]     F2←2×+/EV×(STV[1]÷CRVT*2)×(CRVT×(DATAM[;8]×PRXTD1)+DAT
        AM[;9]×PRXTD2)-CRVS×COSV+(STV[2]×PRXTD1)+STV[3]×PRXTD2
[8]     F3←2×+/EV×(STV[1]÷CRVT*2)×(CRVT×(DATAM[;8]×PRYTD1)+DAT
        AM[;9]×PRYTD2)-CRVS×SINV+(STV[2]×PRYTD1)+STV[3]×PRYTD2
[9]     FV←3 1ρ(F1,F2,F3)
        ∇
```

```
      ∇KITERATE[□]∇
      ∇KITERATE STV;II;CHIS;LNUB;UNUB;IVM;FCNB;STVB;CYC;SIGM
      B
[1]   II←1
[2]   NUVECT STV
[3]   CHIS←(ρDATAM)[1]×MEASM[1;5]*2
[4]   LNUB←(DATAM[;5]×(TAR[8]-TAR[5])÷CRVS)[(⍋DATAM[;5]×(TAR
      [8]-TAR[5])÷CRVS)[1]]
[5]   UNUB←(DATAM[;5]×(TAR[8]+TAR[5])÷CRVS)[(⍒DATAM[;5]×(TAR
      [8]+TAR[5])÷CRVS)[1]]
[6]   SET:CYC←0
[7]    SIGMB←SIGM STV
[8]    IVM←⊞(JACOB STV)
[9]    FCNB←FCN STV
[10]   STVB←STV
[11]   STVA←STV←3ρ((3 1ρSTV)-IVM+.×FCNB)
[12]  TEST:R2←(STV[2]*2)+STV[3]*2
[13]   →(((STV[1]<LNUB)+(STV[1]>UNUB)+(R2>TAR[5]*2)+((SIGM ST
      V)≥SIGM STVB))<1)/CON
[14]   CYC←CYC+1
[15]   STVA←STV←3ρ((3 1ρSTVB)-(2*-CYC)×IVM+.×FCNB)
[16]   →(CYC≤8)/TEST
[17]   STV←STVA←STVB
[18]  CON:NUVECT STV
[19]   II←II+1
[20]   SIGMA←SIGM STV
[21]   →((|SIGMA-SIGMB)<10*¯5)/0
[22]   →(II<REPIT)/SET
      ∇
```

```
      ∇JACOB[□]∇
      ∇JA←JACOB STV;CI;DEL;JA;FJ1;FJ2;DEM
[1]   CI←1
[2]   DEL←(0,0,10*-6)
[3]   JA←3 0ρ0
[4]   FJ1←FCN STV
[5]  JI:DEL←¯1⌽DEL
[6]   DEM←DEL×(100 1 1)
[7]   FJ2←FCN(STV+DEM)
[8]   JA←JA,(FJ2-FJ1)×÷DEM[CI]
[9]   CI←CI+1
[10]  →(3≥CI)/JI
      ∇
```

```
        ∇SSSTM[□]∇
        ∇SSSTM STV;J;RN;K;AV;RNS;N;STVW;FM
[1]     J←1
[2]     SSTM←32 6ρ0
[3]   SRN:RN←10×J
[4]     KITERATE STV
[5]     SSTM[J;]←RN,STVA,SIGMA,J
[6]     J←J+1
[7]     →(J≤12)/SRN
[8]     AV←12ρ0
[9]     AV[1]←SSTM[1;5]<SSTM[2;5]
[10]    AV[12]←SSTM[12;5]<SSTM[11;5]
[11]    K←2
[12]  VEC:AV[K]←(SSTM[K;5]<SSTM[K-1;5])
[13]    AV[K]←AV[K]×SSTM[K;5]<SSTM[K+1;5]
[14]    K←K+1
[15]    →(K<12)/VEC
[16]    →((+/AV)≥1)/SUP
[17]    SSTM[13;]←6ρ0
[18]    →0
[19]  SUP:RNS←AV/12↑,SSTM[;6]
[20]    N←1
[21]  STT:RN←SSTM[RNS[N];1]-5
[22]    STVW←SSTM[RNS[N];2 3 4]
[23]    KITERATE STVW
[24]    SSTM[12+(3×N)-2;]←RN,STVA,SIGMA,12+(3×N)-2
[25]    RN←SSTM[RNS[N];1]
[26]    KITERATE STVW
[27]    SSTM[12+(3×N)-1;]←RN,STVA,SIGMA,12+(3×N)-1
[28]    RN←SSTM[RNS[N];1]+5
[29]    KITERATE STVW
[30]    SSTM[12+3×N;]←RN,STVA,SIGMA,12+3×N
[31]    N←N+1
[32]    →(N≤ρRNS)/STT
[33]    R←(⍋(12+3×ρRNS)↑,SSTM[;5])[1]
[34]    RN←SSTM[R;1]-2.5
[35]    STVW←SSTM[R;2 3 4]
[36]    KITERATE STVW
[37]    FM←3 5ρ0
[38]    FM[1;]←RN,STVA,SIGMA
[39]    RN←SSTM[R;1]
[40]    KITERATE STVW
[41]    FM[2;]←RN,STVA,SIGMA
[42]    RN←SSTM[R;1]+2.5
[43]    KITERATE STVW
[44]    FM[3;]←RN,STVA,SIGMA
[45]    MIN←FM[(⍋,FM[;5])[1];]
        ∇
```

```
        ∇RRNG[□]∇
        ∇RRNG  STV;NUA;TUA;BB;RVV;EVV;RIM;RIMD;RCN;DRCN;RNA;DN;
        DPD;LB;FV;UB
[1]     NUA←(⍉(DATAM[;2]=1)×(DATAM[;3]≤SRCH1[5]))/⍉DATAM[;5]
[2]     TUA←((⍉(DATAM[;2]=1)×(DATAM[;3]≤SRCH1[5]))/⍉DATAM[;3])
        -DATAM[1;3]
[3]     BB←DATAM[1;4]
[4]     RVV←STV[2 3]-SRCH1[3 4]
[5]     EVV←NUA-STV[1]
[6]     DPD←((2○BB)×RVV[1])+(1○BB)×RVV[2]
[7]     □←LB←-((((RVV[1]*2)+RVV[2]*2)*.5)×(ρNUA)-1)+DPD
[8]     □←FV←(TAR[8]÷STV[1])×+/EVV
[9]     □←UB←(ρNUA)×-DPD
[10]    →((LB≤FV)×(FV≤UB)=1)/CONT
[11]    →(FV<LB)/LBL
[12]    →(UB<FV)/UBL
[13] LBL:RN←10
[14] UBL:RN←120
[15]    ' NO INITIAL RANGE SOLUTION POSSIBLE, RN = ';RN;' ASSU
        MED '
[16]    →0
[17] CONT:RIM←(2,(ρNUA))ρ0
[18]    RN←50
[19] CALC:RIMD←RN+TUA×DPD
[20]    RIM[1;]←((RN×2○BB)+TUA×RVV[1])÷RIMD
[21]    RIM[2;]←((RN×1○BB)+TUA×RVV[2])÷RIMD
[22]    RCN←((TAR[8]÷STV[1])×+/EVV)+(RVV[1]×+/RIM[1;])+RVV[2]×
        +/RIM[2;]
[23]    DRCN←-(((RVV[1]*2)+RVV[2]*2)-(DPD*2))×+/TUA÷RIMD*2
[24]    RNA←RN-RCN÷DRCN
[25]    DN←|RNA-RN
[26]    RN←RNA
[27]    →(DN>10*¯4)/CALC
[28]    'NEWTON-RAPHSON SOLN FOR RN = ';RN
        ∇




        ∇RNSEL[□]∇
        ∇RNSEL STV;C
[1]     RNMT←23 2ρ0
[2]     C←1
[3]  RNS:RN←10+5×(C-1)
[4]     RNMT[C;]←RN,SIGM STV
[5]     C←C+1
[6]     →(C≤23)/RNS
[7]     RN←RNMT[(⍋,RNMT[;2])[1];1]
        ∇
```

```
      ∇SHIFT[□]∇
      ∇SHIFT
[1]   'SET XTV,YTV,XSV,YSV'
[2]   IV←,□
[3]   XTV←IV[1]
[4]   YTV←IV[2]
[5]   XSV←IV[3]
[6]   YSV←IV[4]
[7]   'SET N = NO. REPIT.'
[8]   N←,□
[9]   'SET D = INTER PULSE TIME'
[10]  D←,□
[11]  K←0
[12] CHS:X←?150
[13]  Y←?150
[14]  'WHEN X = ';X;' AND Y = ';Y
[15]  DEXACT
[16]  DAPPROX
[17]  'DEXACT = ';DA1
[18]  'DAPPROX = ';DAA;' PERCENT ERROR = ';100×(DAA-DA1)÷DA1
[19]  'DAPPROX2 = ';DAA2;' PERCENT ERROR = ';100×(DAA2-DA1)÷
      DA1
[20]  K←K+1
[21]  →(K≤N)/CHS
      ∇
```

```
      ∇DEXACT[□]∇
      ∇DEXACT
[1]   DA←D
[2]   DA1←D+(÷C)×(((((X-(DA×XSV)-D×XTV)*2)+(Y-(DA×YSV)-D×YTV)
      *2)*.5)-((X*2)+Y*2)*.5
[3]   →((|DA-DA1)<10*¯10)/0
[4]   DA←DA1
[5]   →2
      ∇
```

```
      ∇DAPPROX[□]∇
      ∇DAPPROX
[1]   R←((X*2)+Y*2)*.5
[2]   DAA←D×(C+(XTV×X÷R)+YTV×Y÷R)÷C+(XSV×X÷R)+YSV×Y÷R
[3]   DAA2←D×1+(÷C)×(XTV×X÷R)+(YTV×Y÷R)-((XSV×X÷R)+YSV×Y÷R)
      ∇
```

```
      ∇SHIFTY[□]∇
      ∇SHIFTY
[1]   'SET XTV,YTV,XSV,YSV,X.Y'
[2]   IV←,□
[3]   XTV←IV[1]
[4]   YTV←IV[2]
[5]   XSV←IV[3]
[6]   YSV←IV[4]
[7]   X←IV[5]
[8]   Y←IV[6]
[9]   'SET N = NO. REPIT.'
[10]  N←,□
[11]  'SET D = INTER PULSE TIME'
[12]  D←,□
[13]  K←0
[14] CHS:X←X+.1×XTV-XSV
[15]  Y←Y+.1×YTV-YSV
[16]  'WHEN X = ';X;' AND Y = ';Y
[17]  DEXACT
[18]  DAPPROX
[19]  'DEXACT = ';DA1
[20]  'DAPPROX = ';DAA;' PERCENT ERROR = ';100×(DAA-DA1)÷DA1
[21]  'DAPPROX2 = ';DAA2;' PERCENT ERROR = ';100×(DAA2-DA1)÷
      DA1
[22]  K←K+1
[23]  →(K≤N)/CHS
      ∇
```

APPENDIX B

THE UNIQUENESS PROBLEM FOR PERFECT INFORMATION

If there were no errors in the measurements of bearing or frequency, the fundamental system [B-2], which derives from several measurements of [B-1], would possess a unique solution for all parameters, provided at least three different bearings were included.

$$f_a (C + \vec{r} \cdot \vec{V_T}) = f (C + \vec{r} \cdot \vec{V_S}) \qquad [B-1]$$

$$(C + \vec{r_1} \cdot \vec{V_{S_1}}) f - f_{a_1} \cos B_1 \dot{X}_T - f_{a_1} \sin B_1 \dot{Y}_T = f_{a_1} C$$

$$(C + \vec{r_N} \cdot \vec{V_{S_N}}) f - f_{a_N} \cos B_N \dot{X}_T - f_{a_N} \sin B_N \dot{Y}_T = f_{a_N} C$$

[B-2]

To prove the assertion of a unique solution for the range parameter, it suffices to establish that the system [B-2] has a unique solution for the three parameters f, $\dot{X}_T$, $\dot{Y}_T$. Indeed, geometrical considerations make it clear that any two distinct bearings, B_i and B_j, obtained at times TT_i and TT_j (without error), together with the known values $\dot{X}_T$, $\dot{Y}_T$ will, to all practical purposes, uniquely determine range, the very small indeterminacy introduced by the change of transmission time of the two pulses being negligible.

To prove that B-2 possesses a unique solution, we only have to consider $N=3$ with B_1, B_2 and B_3 all different. Under these circumstances there will be a unique solution so long as the determinant

$$D = \begin{vmatrix} C + \vec{r}_1 \cdot \vec{V}_{S_1} & - f_1 \cos B_1 & - f_1 \sin B_1 \\ C + \vec{r}_2 \cdot \vec{V}_{S_2} & - f_2 \cos B_2 & - f_2 \sin B_2 \\ C + \vec{r}_3 \cdot \vec{V}_{S_3} & - f_3 \cos B_3 & - f_3 \sin B_3 \end{vmatrix} \qquad [B-3]$$

does not vanish.

Equation [B-1] makes it possible to replace the elements of the first column of [B-3] by expressions of the form:

$$\frac{f_i}{f} (C + \vec{r}_i \cdot \vec{V}_T)$$

Then the determinant can be cleared of the f_i by multiplying each one of its rows by the appropriate non-zero factor f/f_i. The following simplified determinant D' thus results. D' = 0 if, and only if, D = 0.

$$D' = \begin{vmatrix} C + \cos B_1 \, \dot{X}_T + \sin B_1 \, \dot{Y}_T & - f \cos B_1 & - f \sin B_1 \\ C + \cos B_2 \, \dot{X}_T + \sin B_2 \, \dot{Y}_T & - f \cos B_2 & - f \sin B_2 \\ C + \cos B_3 \, \dot{X}_T + \sin B_3 \, \dot{Y}_T & - f \cos B_3 & - f \sin B_3 \end{vmatrix} \qquad [B-4]$$

The determinant D' can vanish only if there are constants "a" and "b", so that the third row is the linear sum of "a" times the first row and "b" times the second. Then, the three equations in [B-5] would be satisfied:

$$\sin B_3 = a \sin B_1 + b \sin B_2$$

$$\cos B_3 = a \cos B_1 + b \cos B_2$$

$$C + \cos B_3 \, \dot{X}_T + \sin B_3 \, \dot{Y}_T = (a + b) \qquad\qquad [B-5]$$
$$C + (a \cos B_1 + b \cos B_2) \, \dot{X}_T$$
$$+ (a \sin B_1 + b \sin B_2) \, \dot{Y}_T$$

However, substituting the first two equations into the last yields

$$C = (a+b)C \qquad \text{or} \qquad a+b = 1 \qquad\qquad [B-6]$$

Simultaneously, the first two equations of [B-5] state that the unit vector $\vec{r}_3$ is the linear combination $a\vec{r}_1 + b\vec{r}_2$. However, vectors of the form $a\vec{r}_1 + b\vec{r}_2$, with $a + b = 1$, stretch from the origin in two-dimensional space to points on the line joining the termini of the two unit vectors $\vec{r}_1$ and $\vec{r}_2$. However, it can be seen geometrically that no other vector to that line has length one, except $\vec{r}_1$ and $\vec{r}_2$ themselves. Thus since no two of $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ can coincide, the determinant D cannot vanish, and the expressions in [B-3] yield unique solutions for f, $\dot{X}_T$, $\dot{Y}_T$, as asserted.

DREV R-4199/81 (UNCLASSIFIED)

Research and Development Branch, DND, Canada.
DREV, P.O. Box 880, Courcelette, Que. GOA 1RO

"Algorithms for Tracking a Sound Source Using Frequency and Bearings"
by C. Kent

This report develops some of the analytical support for
algorithms designed to permit one or more mobile searchers to track
a sound source moving in a straight line. The searchers measure
bearing and received frequency to infer range, the true frequency
and velocity components. The measurements of bearing and frequency
are subject to random errors. (U)

CRDV R-4199/81 (NON CLASSIFIE)

Bureau – Recherche et Développement, MDN, Canada.
CRDV, C.P. 880, Courcelette, Qué. GOA 1RO

"Algorithmes basés sur la fréquence et les gisements pcur la poursuite d'une source sonore"
par C. Kent

Dans ce rapport, on développe certains fondements théoriques d'algorithmes conçus pour permettre à une ou plusieurs unités mobiles de recherche de poursuivre une source de bruit qui se déplace en ligne droite. Les unités de recherche mesurent le gisement et la fréquence apparente afin d'estimer la distance, la fréquence réelle et les composantes de la vitesse de la source. On présume que les mesures de gisement et de fréquence sont sujettes à des erreurs aléatoires. (NC)

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